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# What is an Elementary Physical Object? A System-Theoretic Approach

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An exact system-theoretic explanation is given of the intuitive physical conjecture that the notion of elementarity is symmetry-dependent.

## 1. INTRODUCTION

According to a well-known theorem in general systems theory (GST), an arbitrary system in principal can be decomposed into subsystems (Mesarovic and Takahara, 1975).

On the other hand, there is an old, intuitively justified rule that the elementary particles are identified with the irreducible representations of a certain symmetry group. This suggests that elementarity is a notion depending on the supposed symmetries of physical objects (Martin and Spearman, 1970).

After a short review of the basic definitions of GST, I consider the problem of elementarity within the framework of GST.

## 2. SUMMARY OF THE BASIC NOTIONS OF GST

*Definition 1.* A system $S$ is a relation:

$$S \subset X \times Y$$

where

$$X = \bigtimes_{i \in I} V_i$$

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is the input set and

$$Y = \bigtimes_{i \in I} W_i$$

is the output set of the system.

*Definition 2.* A set $C$ and a map $R: (C \times X) \to Y$ are called the state set and the (global) response function if

$$(x, y) \in S \Leftrightarrow (\exists c)[c \in C \,\&\, R(c, x) = y] \tag{1}$$

It can be shown that an arbitrary system has a global response function and a state set.

The state set of the system is defined as (see Fig. 1)

$$C := \{f_c | f_c: X \to Y \,\&\, f_c \subseteq S\} \tag{2}$$

The following is a definition of a system-theoretic notion of the composition of systems that is adequate to the composition of two (interacting) physical systems:

*Definition 3.* Let $S_1$ and $S_2$ be the following two systems:

$$S_1 C (X_1 \times Z_{21}) \times (Y_1 \times Z_{12}), \qquad S_2 C (X_2 \times Z_{12}) \times (Y_2 \times Z_{21})$$

The composition of these two systems is a system

$$S = S_1 * S_2 \subset (X_1 \times X_2) \times (Y_1 \times Y_2)$$

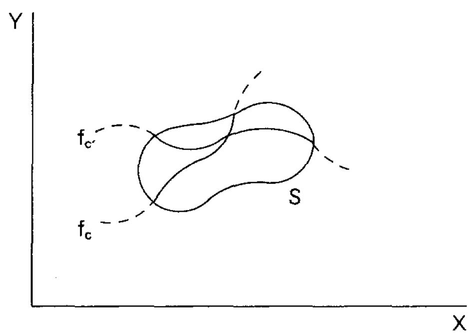


**Fig. 1.**

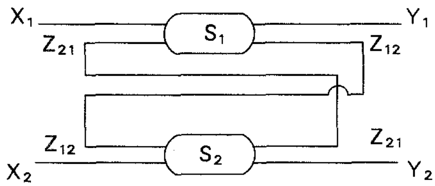


Fig. 2.

defined as follows (Fig. 2):

$$((x_1, x_2), (y_1, y_2)) \in S_1 * S_2$$
$$\Leftrightarrow (\exists z_1)(\exists z_2)[z_1 \in Z_{12} \,\&\, z_2 \in Z_{21}$$
$$\&((z_1, z_2), (y_1, z_1)) \in S_1 \,\&\, ((x_2, z_1), (y_2, z_2)) \in S_2] \tag{3}$$

The question arises of whether an arbitrary system can be decomposed into subsystems. A well-known theorem in GST states that such a decomposition is always possible.

*Theorem 1.* Let $S \subset (X_1 \times X_2) \times (Y_1 \times Y_2)$ be a system. One can find sets $Z_{12}$, $Z_{21}$ and systems

$$S_1 \subset (X_1 \times Z_{21}) \times (Y_1 \times Z_{12}), \qquad S_2 \subset (X_2 \times Z_{12}) \times (Y_2 \times Z_{21})$$

such that $S = S_1 * S_2$ (see Mesarovic and Takahara, 1975).

## 3. SYSTEM WITH SYMMETRIES

The decomposition stated in Theorem 1 is only a theoretical possibility. The picture can be different if one requires that the subsystems have symmetries. The physical motivation of this requirement is that any physical system that really exists should have several basic symmetries.

A symmetric system is defined as follows:

*Definition 4.* Let $S$ be a system $S \subset X \times Y$ and suppose that a group $G$ is acting on $X$ and $Y$ as a transformation group. The system $S$ is said to be $G$-symmetric iff

$$(\forall g \in G)(\forall x \in X)(\forall y \in Y)[(g \cdot x, g \cdot y) \in S \Leftrightarrow (x, y) \in S] \tag{4}$$

where $g \cdot x$ denotes the group action.

Let $C$ be the state set of the system $S$ in the form of (2) and (3). If the system $S$ is $G$-symmetric, one has

$$(x, y)S \Leftrightarrow (g \cdot x, g \cdot y)S \Leftrightarrow (\exists f_{\bar{c}} \in C)[f_{\bar{c}}(g \cdot x) = g \cdot y]$$

From

$$f_{\bar{c}}(g \cdot x) = g \cdot (f_c(x)) = g \cdot y$$

one has

$$f_{\bar{c}} = g \cdot f_c \cdot g^{-1} := U_g f_c \tag{5}$$

This formula defines a canonical $G$-action on the state set.

One needs the construction of the state set of a composite system. Let $S$ be the composition of subsystems $S_1$ and $S_2$. From (3) one has

$$((x_1, x_2), (y_1 y_2))S \Leftrightarrow (\exists Z_1)(\exists Z_2)[((x_1 Z_2)(y_1 z_1)) \in S_1 \\ \&((x_2 z_1), (y_2 z_2)) \in S_2]$$

The state set and the global response function have to satisfy

$$R: (Cx(X_1 \times X_2)) \to Y_1 \times Y_2$$

and

$$\begin{aligned}
&(x_1, x_2, y_1, y_2) \in S \\
&\quad \Leftrightarrow (\exists c \in C)[R(c_1, x_1, x_2) = (y_1, y_2)] \\
&\quad \Leftrightarrow (\exists Z_1)(\exists Z_2)[((x_1, z_2), (y_1, z_1)) \in S \\
&\qquad \&((x_2, z_1), (y_2, z_2)) \in S_2] \\
&\quad \Leftrightarrow (\exists z_1)(\exists z_2)(\exists c_1)(\exists c_2)[R_1(c_1, x_1, z_2) = (y_1, z_1) \\
&\qquad \& R_2(c_2, x_2, z_1) = (y_2, z_2)] \\
&\quad \Leftrightarrow (\exists (c_1, c_2, z_1, z_2))[R((c_1, c_2, z_1, z_2), (x_1, x_2)) \\
&\qquad = (y_1, y_2)]
\end{aligned}$$

From this condition one has:

*Theorem 2.* For a composite system the state set and the global response function are

$$C = C_1 \times C_2 \times Z_1 \times Z_2$$

and

$$\begin{aligned}
&R: (C_1 \times C_2 \times Z_1 \times Z_2) \times (X_1 \times X_2) \to Y_1 \times Y_2 \\
&R((c_1, c_2, z_1, z_2), (x_1, x_2)) \\
&\quad = (\mathrm{pr}_1 \circ R_1(c_1, x_1, z_2), \mathrm{pr}_1 \circ R_2(c_2, x_2, z_1))
\end{aligned}$$

## 4. G-ELEMENTARY SYSTEM

*Definition 5.* A $G$-symmetric system $S$ is said to be $G$-elementary iff there are not $G$-symmetric systems $S_1$ and $S_2$ such that $S = S_1 * S_2$.

*Definition 6.* A $G$-symmetric system is said to be irreducible iff the state set does not contain $G$-invariant subsets, i.e., it is not a set of the form $C = C_1 \times C_2$, where $U_g C_1 \subseteq C_1$ and $U_g C_2 \subseteq C_2$, $\forall g \in G$.

Now the following question arises. When can a $G$-symmetric system be decomposed into $G$-symmetric subsystems?

*Theorem 3.* A $G$-symmetric system $S$ is $G$-elementary if and only if it is $G$-irreducible.

*Proof.* Let $S = S_1 * S_2$ be a decomposition of $S$. The state set of $S$ is

$$C = C_1 \times C_2 \times Z_{12} \times Z_{21}$$

From (5) one has the generated group actions on $C_1$ and $C_2$. Namely for any $(c_2 z_1 z_2) \in C_2 \times Z_{12} \times Z_{21}$, $(c_1 z_1 z_2) \in C_1 \times Z_{12} \times Z_{21}$, and $g \in G$ one has

$$(c_2 z_1 z_2)\, U_g^{(1)}(c_1) = \mathrm{pr}_1(Ug(c_1 c_2 z_1 z_2))$$

$$(c_1 z_1 z_2)\, U_g^{(2)}(c_2) = \mathrm{pr}_2(Ug(c_1 c_2 z_1 z_2))$$

The subsystems $S_1$ and $S_2$ are $G$-symmetric if and only if $(c_2 z_1 z_2)\, U_g^{(1)}$ and $(c_1 z_1 z_2)\, U_g^{(2)}$ are independent of $(C_2 Z_1 Z_2)$ and $(c_1 z_1 z_2)$, respectively. In that case one has

$$U_g(C_1 C_2 Z_1 Z_2) = (\chi_g^{(1)}(G), \chi_g^{(2)}(z_2), \chi_g(Z_1 Z_2))$$

where

$$\chi_g^{(1)}: C_1 \to C_1, \qquad \chi_g^{(2)}: C_2 \to C_2$$

$$\chi_g: Z_{12} \times Z_{21} \to Z_{12} \times Z_{21}$$

that is, the system $S$ cannot be $G$-irreducible. ■

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